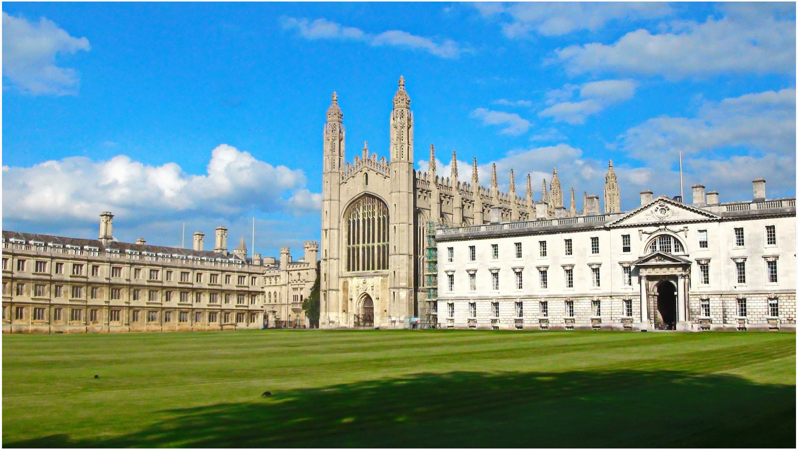
Complexity Theory

Lecture 1: Introduction and motivation

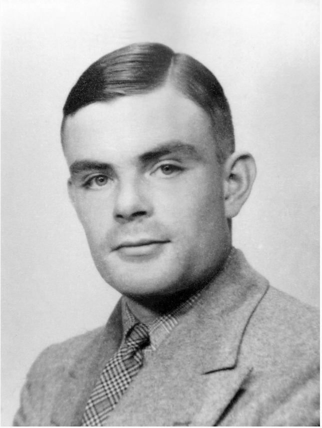
Tom Gur

http://www.cl.cam.ac.uk/teaching/2324/Complexity

The story starts here in Cambridge...



Alan Turing and Computation Theory



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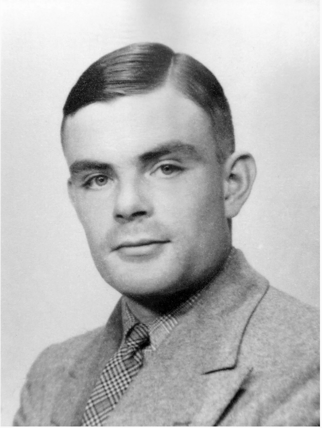

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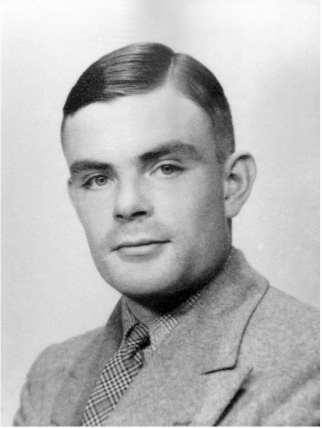

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Or is it...

Numbers and Scale

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$2^{1000000}$ complexity of an exponential-time algorithm on a small input...

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So let's start!

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But, what is the complexity of the sorting problem?

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- $f = O(g)$, if there is an $n_0 \in \mathbb{N}$ and a constant c such that for all $n > n_0$, $f(n) \leq cg(n)$;
- $f = \Omega(g)$, if there is an $n_0 \in \mathbb{N}$ and a constant c such that for all $n > n_0$, $f(n) \geq cg(n)$.
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Usually, O is used for upper bounds and Ω for lower bounds.

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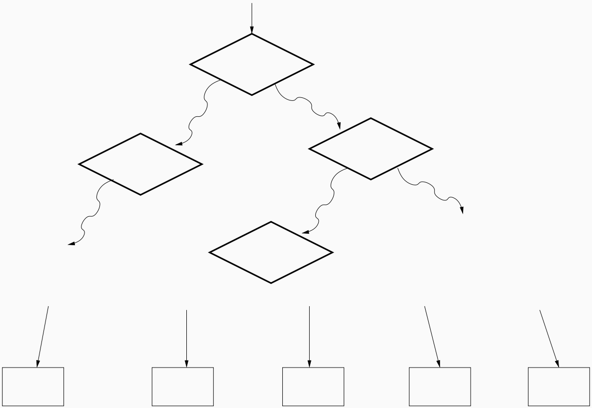
Sorting is a rare example where known upper and lower bounds match.

Lower Bound on Sorting

An algorithm A sorting a list of n distinct numbers $a_1, \ldots, a_n$.



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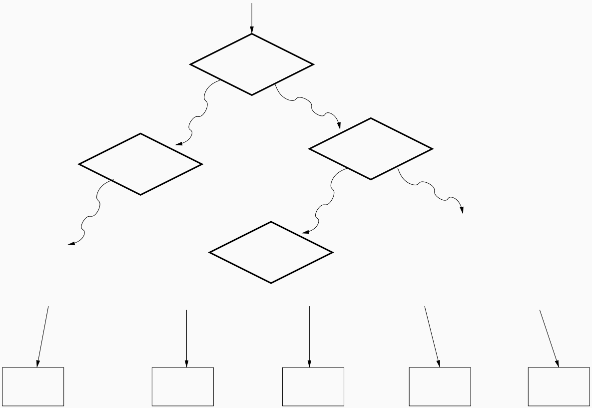

To work for all permutations of the input list, the tree must have at least $n!$ leaves and therefore height at least $\log_2(n!) = \theta(n \log n)$.

Travelling Salesman

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- V — a set of nodes.
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Find an ordering $v_1, \ldots, v_n$ of V for which the total cost:

$$c(v_n, v_1) + \sum_{i=1}^{n-1} c(v_i, v_{i+1})$$

is the smallest possible.

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Between these two is the chasm of our ignorance.

Textbooks

The main texts for the course are:

Computational Complexity.
Christos H. Papadimitriou.

Introduction to the Theory of Computation.
Michael Sipser.

Outline

A rough lecture-by-lecture guide, with relevant sections from the text by Papadimitriou (or Sipser, where marked with an S).

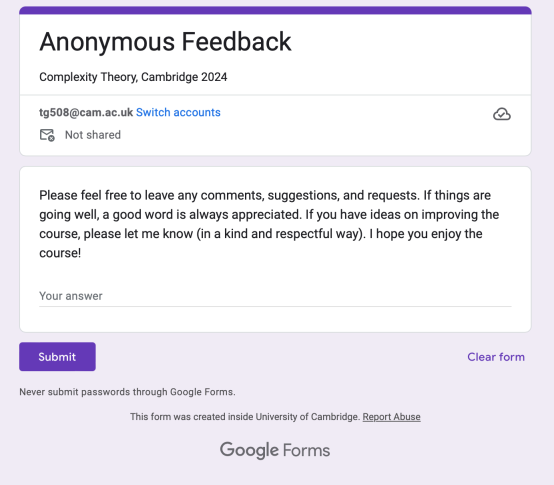
- **Algorithms and problems.** 1.1–1.3.
- **Time and space.** 2.1–2.5, 2.7.
- **Time Complexity classes.** 7.1, S7.2.
- **Nondeterminism.** 2.7, 9.1, S7.3.
- **NP-completeness.** 8.1–8.2, 9.2.
- **Graph-theoretic problems.** 9.3

Outline

- **Sets, numbers and scheduling.** 9.4
- **coNP.** 10.1–10.2.
- **Cryptographic complexity.** 12.1–12.2.
- **Space Complexity** 7.1, 7.3, S8.1.
- **Hierarchy** 7.2, S9.1.
- **Quantum Complexity** 20 [Arora-Barak]

Anonymous feedback

Let me know what works and what doesn't. Complexity theory is beautiful – let's enjoy and get the most out of it!


Anonymous Feedback

Complexity Theory, Cambridge 2024

tg508@cam.ac.uk Switch accounts

Not shared

Please feel free to leave any comments, suggestions, and requests. If things are going well, a good word is always appreciated. If you have ideas on improving the course, please let me know (in a kind and respectful way). I hope you enjoy the course!

Your answer

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Questions?